Calculus II - Day 21

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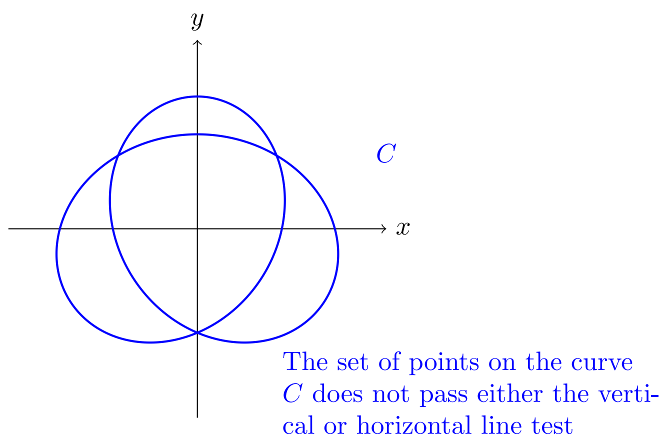
Goals for today

- Plot parametric curves and turn parametric equations into Cartesian equations (and vice versa)
- Use the Chain Rule to compute derivatives of parametric curves
- Compute the arc length of parametric curves

Unfortunately: not every plane curve can be expressed as a function $y = f(x)$ or $x = g(y)$.



To get around this, we parametrize the x- and y- coordinates as functions of a third variable t. $x = f(t)$, and $y = g(t)$

- Think of t as representing time: "at time t" the point on the curve is $(x, y) = (f(t), g(t))$

Example

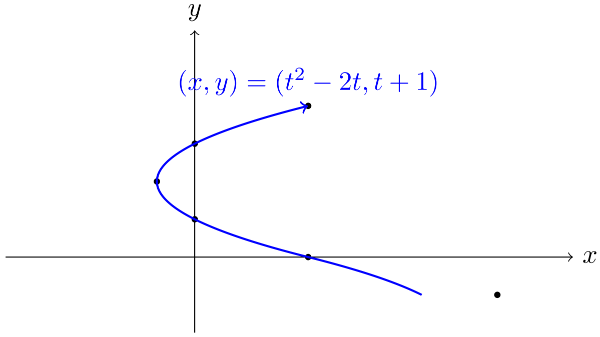
Sketch and identify the curve defined parametrically by the equations

$$x = t^2 - 2t$$

$$y = t + 1$$

Plot points by plugging in values for t:

t	$x = t^2 - 2t$	$y = t + 1$
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4



As t increases, we move along the curve in the positive x direction, so the curve should also have arrows moving along the curve.

This curve passes the horizontal line test, so perhaps it can be expressed as a function $x = h(y)$, independent of t.

$$x = t^2 - 2t$$

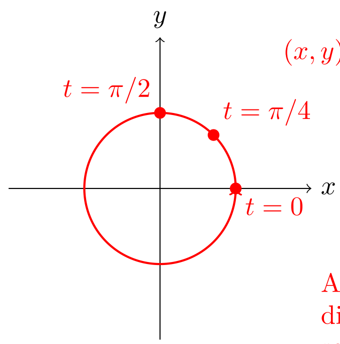
$$y = t + 1$$
$$t = y - 1$$

$$x = (y-1)^2 - 2(y-1)$$
$$x = y^2 - 4y + 3, \text{ a parabola}$$

Problem: this isn't always possible.

Example

Let $x = \cos(t)$ and $y = \sin(t)$, $0 \le t \le 2\pi$.



(x, y) = (cos(t), sin(t))

As t increases, we move along the curve in the indicated direction, so the curve should also have arrows moving along the curve.

To eliminate the parameter, recall that $\cos^2(t) + \sin^2(t) = 1$ ($x^2 + y^2$)

$$\boxed{x^2 + y^2 = 1}$$

In general, the circle of radius r centered at (a, b) is:

Parametric:

$$x = a + r\cos(t)$$
$$y = b + r\sin(t)$$
$$0 \le t \le 2\pi$$

Cartesian:

$$(x - a)^2 + (y - b)^2 = r^2$$

Parametrizations are not unique

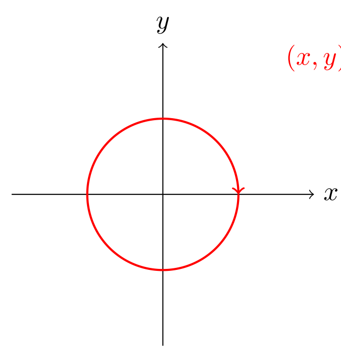
Example:

$$x = \sin(2t)$$

$$y = \cos(2t)$$

$$0 \le t \le 2\pi$$

These equations parametrize two rotations around the unit circle clockwise starting at $(0, 1)$.



(x, y) = (sin(2t), cos(2t))

Same curve, different parametrization.
Q: Given a Cartesian equation, how can it be written in parametric form?
Example: Parameterize the line $y = 2x - 4$ on the interval $1 \le x \le 4$.
Since this is a function of x, the simplest way is:

$$x = t, \quad y = 2t - 4, \quad 1 \le t \le 4$$

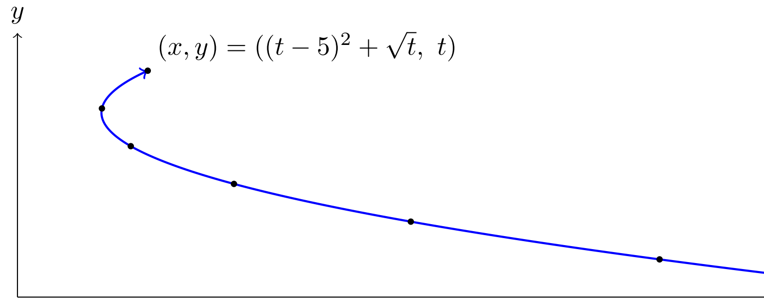
Example

Parametrize and graph the curve described by

$$x = (y-5)^2 + \sqrt{y}$$

Parametrization

t	$y=t$	$x=(t-5)^2+\sqrt{t}$
0	0	25
1	1	17
2	2	$9+\sqrt{2}$
3	3	$4+\sqrt{3}$
4	4	3
5	5	$\sqrt{5}$
6	6	$1+\sqrt{6}$



Derivatives of Parametric Curves:
Let $x = f(t)$ and $y = g(t)$. What is the slope of the parametric curve at time t?
We want: $\frac{dy}{dx}$, but y might not be a function of x.

$$g'(t) = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot f'(t)$$
$$\Rightarrow \frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

This is the formula for the slope in terms of t.

Theorem: If $x = f(t), y = g(t)$, then the slope of the curve $(x, y) = (f(t), g(t))$ is

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} \quad \text{as long as} \quad f'(t) \neq 0.$$

Example: Let C be the curve parameterized by $x = t^2$, $y = t^3 - 3t$. Find all tangent lines to C at the point $(x, y) = (3, 0)$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-3}{2t} = \frac{3}{2}\left(\frac{t-1}{t}\right).$$

We need to know at what time(s) (value(s) of t) this curve passes through $(3, 0)$.
First, solve for $x = t^2 = 3$, which gives $t = \pm\sqrt{3}$. Next, for $y = t^3 - 3t = 0$, plug in $\pm\sqrt{3}$:

$$\text{For } t = \sqrt{3}: \quad (\sqrt{3})^3 - 3\sqrt{3} = 3\sqrt{3} - 3\sqrt{3} = 0 \quad (\checkmark)$$
$$\text{For } t = -\sqrt{3}: \quad (-\sqrt{3})^3 - 3(-\sqrt{3}) = -3\sqrt{3} + 3\sqrt{3} = 0 \quad (\checkmark)$$

To get the tangent lines, start by plugging in $t = \pm\sqrt{3}$ into the derivative equation:

For $t = \sqrt{3}$,

$$\left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \frac{3}{2}\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)$$

$$= \frac{3\sqrt{3}}{2} - \frac{3}{\sqrt{3}} \times \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

Equation: $y - y_0 = m(x - x_0)$, where $x_0 = 3, y_0 = 0, m = \sqrt{3}$:

$$y - 0 = \sqrt{3}(x-3) \quad \Rightarrow \quad y = \sqrt{3}x - 3\sqrt{3}.$$

For $t = -\sqrt{3}$,

$$= \frac{3}{2}\left(\frac{-\sqrt{3}-1}{-\sqrt{3}}\right) = \frac{-3\sqrt{3}}{2} + \frac{3}{2\sqrt{3}} = \frac{-\sqrt{3}}{2}$$

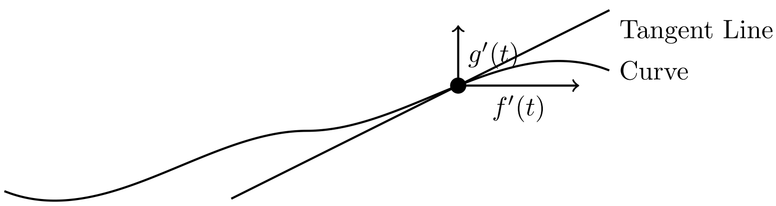
Equation: $y - y_0 = m(x - x_0)$, where $x_0 = 3, y_0 = 0, m = -\sqrt{3}$:

$$y - 0 = -\sqrt{3}(x-3) \quad \Rightarrow \quad y = -\sqrt{3}x + 3\sqrt{3}.$$

Arc Length of Parametric Curves

Let $(x, y) = (f(t), g(t))$. How fast are we traveling along the curve at time t?

Illustration



Speed Along the Tangent Line

$$\text{Speed along tangent line: } \sqrt{\big(f'(t)\big)^2 + \big(g'(t)\big)^2}$$

Parametric Arc Length Formula

$$L = \int_{t=a}^{t=b} \sqrt{\big(f'(t)\big)^2 + \big(g'(t)\big)^2}\, dt$$

Example: Circumference of a Circle of Radius r

Parametric equations for the circle:

$$x = r\cos(t) = f(t), \quad y = r\sin(t) = g(t)$$

Compute derivatives:

$$\big(f'(t)\big)^2 = \big(-r\sin(t)\big)^2 = r^2\sin^2(t), \quad \big(g'(t)\big)^2 = \big(r\cos(t)\big)^2 = r^2\cos^2(t)$$

Substitute into arc length formula:

$$L = \int_0^{2\pi} \sqrt{\big(f'(t)\big)^2 + \big(g'(t)\big)^2}\, dt = \int_0^{2\pi} \sqrt{r^2\sin^2(t) + r^2\cos^2(t)}\, dt$$

Simplify using $\sin^2(t) + \cos^2(t) = 1$:

$$L = \int_0^{2\pi} \sqrt{r^2}\, dt = \int_0^{2\pi} r\, dt$$

Evaluate the integral:

$$L = r \cdot t\Big|_0^{2\pi} = 2\pi r$$

$$\boxed{2\pi r}$$